Continuous Symmetries of Lattice Conformal Field Theories and their $\mathbb{Z}_2$-Orbifolds

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February 1, 2008

Abstract

Following on from a general observation in an earlier paper [10], we consider the continuous symmetries of a certain class of conformal field theories constructed from lattices and their reflection-twisted orbifolds. It is shown that the naive expectation that the only such (inner) symmetries are generated by the modes of the vertex operators corresponding to the states of unit conformal weight obtains, and a criterion for this expectation to hold in general is proposed.

1 Introduction

The identification of the automorphism group of a conformal field theory, while being clearly of import in its own right as a guide towards an understanding of the general structure of the theory, is an essential tool in the classification program. It has been demonstrated in [4, 2] that the FKS lattice theories and their reflection-twisted orbifolds [3] (see section 2 for definitions) form a crucial element in the classification of self-dual conformal field theories. Such a classification needs to be performed separately from any mainstream approach involving consideration of fusion rules [14], since the self-dual theories are trivial from this point of view, and physically is relevant in heterotic string theory [11]. Steps towards this classification have been accomplished from two opposite approaches. On the one hand, Schellekens has restricted the possible algebras which correspond to the weight one states in the theories at central charge 24 [12, 13], while, on the other hand, constructions of theories which exhibit these algebras have been accomplished [3]. The only theories constructed so far are the FKS lattice theories and their reflection-twisted orbifolds. It is generally believed that the remaining theories are orbifolds of some form (either of the FKS theories themselves, or of orbifolds of these theories), and investigations are proceeding along these lines (see *e.g.* [10]). As a result, the identification of the automorphism groups of the FKS theories and their only known consistent orbifolds (the reflection-twisted theories) are of import in as much as enabling a full classification of all orbifolds which may be obtained.

Some comments on the discrete part of the automorphism group of these theories are contained in [10]. In this letter, we discuss the continuous symmetries of the theories.

We begin in section 2 by briefly describing the FKS lattice conformal field theories $\mathcal{H}(\Lambda)$ and their reflection-twisted orbifolds $\widetilde{\mathcal{H}}(\Lambda)$ (Λ an even lattice). This is simply a summary of the relevant parts of previous work [3]. We then summarize (and slightly refine) in section 3 the comments made in [10] on continuous symmetries in general, before proceeding to apply our considerations in sections 4 and 5 to $\mathcal{H}(\Lambda)$ and $\widetilde{\mathcal{H}}(\Lambda)$ respectively.

Our conclusions are presented in section 6.

2 The conformal field theories $\mathcal{H}(\Lambda)$ and $\widetilde{\mathcal{H}}(\Lambda)$

Throughout this paper, the term conformal field theory shall be taken to mean bosonic chiral meromorphic conformal field theory. $\mathcal{H}$ is a (bosonic chiral meromorphic) conformal field theory if [8] $\mathcal{H}$ is a Hilbert space equipped with a set of linear *vertex* operators

$V(\psi,z):\mathcal{H}\to\mathcal{H}$ ($\psi\in\mathcal{H}$, $z\in\mathbf{C}$) such that for $\psi,\phi\in\mathcal{H}$

$$V(\psi,z)V(\phi,w)=V(\phi,w)V(\psi,z)\,, \tag{1}$$

in the sense of analytic continuation in the complex variables z and w of the meromorphic matrix elements we may obtain from either side (the left hand side, for example, being strictly only defined for $|z|>|w|$). It may be shown (and this is often taken as an axiom) that

$$V(\psi,z)V(\phi,w)=V\left(V(\psi,z-w)\phi,w\right)\,, \tag{2}$$

again with suitable analytic continuation. This is the so-called *operator product expansion* or duality relation. There are also preferred states $|0\rangle,\psi_L\in\mathcal{H}$ such that

$$V(\psi_L,z)\equiv\sum_n L_n z^{-n-2}\,, \tag{3}$$

where

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}m(m^2-1)\delta_{m,-n} \tag{4}$$

for some *central charge* c, and $|0\rangle$ is an SU(1,1) invariant state (annihilated by $L_{\pm1},L_0$) such that

$$V(\psi,z)|0\rangle=e^{zL_{-1}}\psi\,. \tag{5}$$

The space may be decomposed into eigenstates of L_0, the eigenvalues being known as the *conformal weights*. In general, for ψ of conformal weight h, we write

$$V(\psi,z)\equiv\sum_n V(\psi)_n z^{-n-h}\,, \tag{6}$$

and we find that the action of $V_n(\psi)$ shifts the conformal weight of a state by n. Fuller details and consequences of this set of axioms may be found in [2].

We now define a specific class of conformal field theories which will be of relevance in this paper.

Suppose we are given a d-dimensional even (Euclidean) lattice Λ. We shall define a conformal field theory denoted $\mathcal{H}(\Lambda)$, the FKS lattice theory referred to in the introduction.

We introduce a set of bosonic creation and annihilation operators a^i_n, $n\in\mathbf{Z}$, $1\le i\le d$, such that

$$\begin{aligned}[a^i_m,a^j_n] &= m\delta^{ij}\delta_{m,-n}\\ {a^i_m}^\dagger &= a^i_{-m}\,.\end{aligned} \tag{7}$$

Let us denote a_0^i by p^i (the momentum operator) and also introduce an operator $q^i(= q^{i\dagger})$ such that $[q^i, p^j] = i\delta^{ij}$. The Hilbert space of $\mathcal{H}(\Lambda)$ is composed of all linear combinations of states of the form

$$\prod_{a=1}^{N} a_{-n_a}^{i_a} |\lambda\rangle \,, \tag{8}$$

where $n_a \in \mathbf{Z}_+$ and $\lambda \in \Lambda$ with $a_n^i|\lambda\rangle = 0$ for $n > 0$ and $p^i|\lambda\rangle = \lambda^i|\lambda\rangle$. Set

$$X^i(z) = q - ip \log z + i \sum_{n \neq 0} \frac{a_n^i}{n} z^{-n} \,, \tag{9}$$

and then

$$V\left(\prod_{a=1}^{N} a_{-n_a}^{i_a} |\lambda\rangle, z\right) =: \prod_{a=1}^{N} \frac{1}{(n_a - 1)!} \frac{d^{n_a-1} X^{i_a}(z)}{dz^{n_a-1}} e^{i\lambda \cdot X(z)} : \sigma_\lambda \,, \tag{10}$$

where the σ_λ are a set of cocycle operators satisfying

$$\sigma_\lambda \sigma_\mu = (-1)^{\lambda\cdot\mu} \sigma_\mu \sigma_\lambda \tag{11}$$

for λ, $\mu \in \Lambda$, and we use the usual normal ordering convention on the oscillators. This is sufficient (see *e.g.* [3]) to make $\mathcal{H}(\Lambda)$ into a consistent conformal field theory (with central charge d and $\psi_L = \frac{1}{2} a_{-1} \cdot a_{-1}|0\rangle$). The conformal weight of the state (8) is simply $\sum_{a=1}^{N} n_a + \frac{1}{2}\lambda^2$. Physically, this theory represents a bosonic string propagating on the torus $\mathbf{R}^d/\Lambda$.

Now, the lattice clearly admits a reflection symmetry $\lambda \mapsto -\lambda$, and this trivially lifts to an involution θ of $\mathcal{H}(\Lambda)$ ($\theta a_n^i \theta^{-1} = -a_n^i$, $\theta|\lambda\rangle = |-\lambda\rangle$). Let $\mathcal{H}(\Lambda)_+$ be the sub-conformal field theory on which $\theta = 1$.

Suppose now that the dimension d of the lattice is a multiple of 8. We construct [2, 6] a meromorphic representation $\mathcal{H}_T(\Lambda)_+$ of $\mathcal{H}(\Lambda)_+$ as follows. The Hilbert space is built up from a ground state of conformal dimension $\frac{d}{16}$ (which forms an irreducible representation space for a set of gamma matrices γ_λ, $\lambda \in \Lambda$, satisfying $\gamma_\lambda\gamma_\mu = (-1)^{\lambda\cdot\mu}\gamma_\mu\gamma_\lambda$ – *c.f.* (11)) by the action of an odd or even number (according as d is an odd or an even multiple of 8) of half-integrally graded bosonic creation and annihilation operators c_r^i, $r \in \mathbf{Z} + \frac{1}{2}$, $1 \le i \le d$, such that

$$\begin{aligned} [c_r^i, c_s^j] &= r\delta^{ij}\delta_{r,-s} \\ c_r^{i\,\dagger} &= c_{-r}^i \,, \end{aligned} \tag{12}$$

(c^i_r annihilates the ground state for $r > 0$). Vertex operators $V_T(\psi, z)$, $\psi \in \mathcal{H}(\Lambda)_+$ may be defined [3] such that they form a representation of $\mathcal{H}(\Lambda)_+$, *i.e.*

$$V_T(\psi, z)V_T(\phi, w) = V_T\left(V(\psi, z-w)\phi, w\right)\,, \tag{13}$$

c.f. (2). In [3], it is then shown how one may define vertex operators $W(\chi, z)$ and $\overline{W}(\chi, z)$ corresponding to states $\chi \in \mathcal{H}_T(\Lambda)_+$ such that

$$\mathcal{V}\left((\psi, \chi), z\right) = \begin{pmatrix} V(\psi, z) & \overline{W}(\chi, z) \\ W(\chi, z) & V_T(\psi, z) \end{pmatrix} \tag{14}$$

equips the space $\widetilde{\mathcal{H}}(\Lambda) \equiv \mathcal{H}(\Lambda)_+ \oplus \mathcal{H}_T(\Lambda)_+$ with the structure of a conformal field theory, provided that $\sqrt{2}\Lambda^*$ is even (also a necessary condition [9]).

3 Continuous (inner) automorphisms of meromorphic conformal field theories

It is a widely believed result that the continuous symmetries of such a conformal field theory are fully accounted for by the well-known Lie algebra defined by the zero modes of the vertex operators corresponding to the states of conformal weight one. However, there are clear counter examples to this.

Suppose Λ is an even lattice of dimension d, as in the previous section. We define the rank of Λ to be the dimension of the space it spans within this d dimensional space. Consider the extreme case in which Λ has rank zero, *i.e.* set all vectors in the lattice to be identically zero. $\mathcal{H}(\Lambda)$ is then simply the Hilbert space obtained by acting on the vacuum with d commuting sets of creation and annihilation operators a^i_n, and clearly has an $\mathrm{O}(d)$ symmetry group (given by $a^i_n \mapsto R^i{}_j a^j_n$). $\mathcal{H}(\Lambda)_+$ also inherits this $\mathrm{O}(d)$ symmetry group, but there are no states of conformal weight one (the states $a_{-1}|0\rangle$ having been projected out).

In general, let θ be a continuous symmetry of a conformal field theory $\mathcal{H}$, *i.e.*

$$e^{a\theta}V(\psi, z)e^{-a\theta} = V(e^{a\theta}\psi, z)\,, \tag{15}$$

for all $\psi \in \mathcal{H}$ and $a \in \mathbf{R}$, or

$$[\theta, V(\psi, z)] = V(\theta\psi, z)\,. \tag{16}$$

Suppose further that θ is "inner", *i.e.* it can be written in terms of the vertex operators of the theory. Since duality (2) allows us to reduce products of vertex operators, and

any automorphism leaves the Virasoro generators invariant [9] (and hence the conformal weights), we write $\theta = V(\psi_\theta)_0$ for some state $\psi_\theta \in \mathcal{H}$. Expand $\psi_\theta = \sum_{n\geq 1} \psi_n$, where ψ_n is of conformal weight n (we exclude the case $n = 0$ as it simply gives a trivial addition of a constant to θ).

Now, again using the invariance of the Virasoro generators under the automorphism, we must have $[L_{-1}, \theta] = 0$, *i.e.*

$$\sum_{n\geq 1} (n-1) V_{-1}(\psi_n) = 0 \,, \tag{17}$$

from the relation

$$[L_{-1}, V(\psi, z)] = V(L_{-1}\psi, z) = \frac{d}{dz} V(\psi, z) \,, \tag{18}$$

(equivalent to (5)). Suppose that $V_{-1}(\psi) = 0 \Rightarrow \psi = \lambda|0\rangle$ for some $\lambda \in \mathbf{C}$ ($\psi_1 = V_{-1}(\psi_1)|0\rangle$ and $V(|0\rangle) = 1$). Then we may deduce that $\psi_n = 0$ for $n \geq 2$, as required. However, the assumption is too strong. For example, in the theory $\mathcal{H}(\Lambda)$, $V_{-1}(a_{-2}|0\rangle) = 0$. We notice that the relation (18) gives $V_{-1}(L_{-1}\psi_n) = -(n-1)V_{-1}(\psi_n)$ (which accounts for the vanishing of V_{-1} in our example), and we absorb this freedom by using it to redefine the ψ_n to be quasi-primary, *i.e.* annihilated by L_1 (the weight one states are automatically quasi-primary, so there is no problem at $n = 1$). Then our new assumption is that, for ψ quasi-primary, $V_{-1}(\psi) = 0 \Rightarrow \psi = \lambda|0\rangle$ for some $\lambda \in \mathbf{C}$. This gives us the required result, and that it is reasonable will become clear in the following sections where we demonstrate that it holds for both $\mathcal{H}(\Lambda)$ and $\widetilde{\mathcal{H}}(\Lambda)$.

Note that another way of phrasing our condition on the conformal field theory is that, if ψ is quasi-primary, $V_0(\psi)$ determines ψ uniquely up to states of conformal weight one (see appendix A). We define a conformal field theory to be *deterministic* if it satisfies this criterion. Thus, for a deterministic conformal field theory, the continuous inner automorphisms are simply generated by the states of conformal weight one.

In the example of $\mathcal{H}(\Lambda)_+$ given above, the theory is not deterministic as there are redundant operators, *i.e.* the momentum operator is identically zero, and so it is trivial to see that there are (quasi-primary) states with vanishing zero mode.

Conversely, suppose that $\mathcal{H}$ is not deterministic, *i.e.* suppose we have a state ψ (taken to consist only of states of weight at least 2 without loss of generality) such that

$$V_{-1}((L_0 - 1)\psi) \equiv [L_{-1}, V_0(\psi)] = 0 \,. \tag{19}$$

Consider $[V_0(\psi), V(\phi, w)]$ for some state $\phi \in \mathcal{H}$. Applying this to the vacuum, and using

(5) we obtain

$$V_0(\psi)e^{wL_{-1}}\phi + V(\phi,w)V_0(\psi)|0\rangle\,. \tag{20}$$

But, from (19), L_{-1} commutes with $V_0(\psi)$, and also $V_0(\psi)|0\rangle = 0$ ($V_0(\psi)|0\rangle = \gamma|0\rangle$, $\gamma \in \mathbf{C}$, where $\gamma = \langle 0|V(z^{-L_0}\psi,z)|0\rangle = \langle 0|e^{zL_{-1}}z^{-L_0}|\psi\rangle = 0$). Then we see that $[V_0(\psi), V(\phi,w)]$ has the same action on the vacuum as $V(V_0(\psi)\phi,w)$, and so, by the uniqueness theorem [8] (*i.e.* that if $W(z)$ has the same action on the vacuum as $V(\rho,z)$ and is local with respect to the set of vertex operators then $W(z) = V(\rho,z)$), we deduce that

$$[V_0(\psi), V(\phi,w)] = V(V_0(\psi)\phi,w)\,, \tag{21}$$

as required. Note however that there is no guarantee that the automorphism will be non-trivial.

4 Determinism of the theories $\mathcal{H}(\Lambda)$

In this and the following section, we demonstrate that the property of determinism holds for the known self-dual theories, and thus that it is a sensible condition to impose.

We consider a state $\psi \in \mathcal{H}(\Lambda)$ such that $V_0(\psi) = 0$. (Note that we will assume that the rank of Λ is equal to its dimension, so that we will always be able to distinguish the operators $a_0^i \equiv p^i$ from zero.) First decompose ψ according to its momentum, *i.e.* write

$$\psi = \sum_{\lambda\in\Lambda}\sum_{s\in I_\lambda}\prod_{a=1}^{N_\lambda^s} a^{i_{as}^\lambda}_{-n_{as}^\lambda}|\lambda\rangle \equiv \sum_{\lambda\in\Lambda}\psi_\lambda\,. \tag{22}$$

Clearly, we must have $V_0(\psi_\lambda) = 0$ for all $\lambda \in \Lambda$, since $[p, V(\psi_\lambda)] = \lambda V(\psi_\lambda)$. Let us initially consider the case $\lambda = 0$ for simplicity.

Hence, if we decompose ψ_0 into states

$$\psi_0^{n_1\cdots n_d} = \sum_\alpha \rho_1^\alpha\cdots\rho_d^\alpha|0\rangle\,, \tag{23}$$

where ρ_i^α is a set of n_i creation operators with vectorial index i, we see that we must have $V_0(\psi_0^{n_1\cdots n_d}) = 0$ for all $\{n_i\}$.

First note that, as observed in section 3, we are able to rewrite our states by acting with L_{-1}. Rather than considering quasi-primary states, as we did earlier, we show that other conditions may be imposed. All we have to do is simply specify the form of the state sufficiently strongly to absorb all of the ambiguity afforded by the L_{-1} freedom.

For example, suppose that $V_0(\psi) = 0$ and that ψ is of some such specific form, *i.e.* it is a linear combination of states whose L_{-1} descendants span the space and are linearly independent (note that $L_{-1}\phi = L_{-1}\chi \Rightarrow V(L_{-1}(\phi - \chi), z) \equiv \frac{d}{dz} V(\phi - \chi, z) = 0 \Rightarrow \phi - \chi = \mu|0\rangle$ for some scalar μ). Then we are able to use the L_{-1} freedom to define a quasi-primary state $\widehat{\psi}$ such that $V_0(\widehat{\psi}) = V_0(\psi)$ $(= 0)$. Note that the form of the definition is such that $\psi = 0$ if and only if $\widehat{\psi} = 0$ (we must use the linear independence referred to above). Now if $V_0(\widehat{\psi}) = 0 \Rightarrow \widehat{\psi} = \gamma|0\rangle$ for $\widehat{\psi}$ quasi-primary, then $\psi = \gamma|0\rangle$ as required. The converse argument holds, and we see that it is possible to prove the result using some more convenient means of fixing up the freedom.

Now, going back to our particular case, we have $V_0(\psi_0^{n_1 \cdots n_d}) = 0$. Suppose that at least two of the n_i are non-zero (n_1 and n_2 without loss of generality). Set $\rho_2^\alpha \cdots \rho_d^\alpha = \phi^\alpha$. Then we have

$$0 = V_0 \left(\sum_\alpha \rho_1^\alpha \phi^\alpha |0\rangle \right) = \sum_n V_{-n}(\rho_1^\alpha |0\rangle) V_n(\phi^\alpha |0\rangle) \,, \tag{24}$$

(no normal ordering being necessary since the operators commute). Because of the form of the vertex operators, we then see that we must, in particular, have either $V_0(\rho_1^\alpha|0\rangle) = 0$ or $V_0(\phi^\alpha|0\rangle) = 0$. We would then have an inductive step, provided that the restriction chosen to absorb the L_{-1} freedom carried over from $\psi_0^{n_1 \cdots n_d}$ to both $\rho_1^\alpha|0\rangle$ and $\phi^\alpha|0\rangle$.

It is possible to choose a set of states whose L_{-1} descendants span the space in such a way, for example by requiring any non-trivial ρ_i^α to include at least one a_{-1}^i. Though the descendants of such a set are not necessarily linearly independent, a subset of them can be chosen such that this holds (*e.g.* in the case of two oscillators we choose $\{a_{-1}^i a_{-2n-1}^i\}$).

Thus, we have reduced the problem to one of verifying the result in the case of a one-dimensional lattice (at least for the zero-momentum-sector states).

The case of one creation oscillator is trivial.

Let us consider the case of 2 oscillators.

We choose as a set of basis states $\{a_{-n}a_{-n}|0\rangle\}$ (dropping vectorial indices, as we are restricting to one-dimension). (N.B. The action of L_{-1} of course keeps us inside the two-oscillator subspace!) As a simple check that these are sufficient, consider the generating function for the number of states at each level. The states themselves give

$$p(x) = \sum_{n=1}^{\infty} x^{2n} = \frac{x^2}{1 - x^2} \,, \tag{25}$$

and therefore the states and their L_{-1} descendants give, assuming linear independence (which is simply checked:

We already have shown that $L_{-1}\phi = L_{-1}\chi \Rightarrow \phi - \chi = \mu|0\rangle$ for some scalar μ. Therefore, we only have to show that $a_{-n}a_{-n}|0\rangle$ is not an L_{-1} descendant of the higher states.

Suppose the converse. Then we can write

$$a_{-n}a_{-n}|0\rangle = L_{-1}\left(\lambda_1 a_{-1}a_{-(2n-2)}|0\rangle + \cdots + \lambda_{n-1}a_{-(n-1)}a_{-n}|0\rangle\right) . \tag{26}$$

Then

$$\begin{aligned} a_{-n}a_{-n}|0\rangle &= \lambda_1 a_{-2}a_{-(2n-2)}|0\rangle + (2n-2)\lambda_1 a_{-1}a_{-(2n-1)}|0\rangle + \cdots && (27) \\ &\quad + (n-1)\lambda_{n-1}a_{-n}a_{-n}|0\rangle + n\lambda_{n-1}a_{-(n-1)}a_{-(n+1)}|0\rangle . && (28) \end{aligned}$$

Hence we deduce successively $\lambda_1 = 0$, $\lambda_2 = 0$, …, $\lambda_{n-1} = 0$, and also $(n-1)\lambda_{n-1} = 1$, giving the required contradiction.),

$$q(x) = \frac{p(x)}{1-x} = \frac{x^2}{(1-x)(1-x^2)} . \tag{29}$$

Now, the two-oscillator states at level $2n$ are given by $a_{-1}a_{-(2n-1)}|0\rangle$, $a_{-2}a_{-(2n-2)}|0\rangle$, …, $a_{-n}a_{-n}|0\rangle$, with a similar set for odd levels. Hence, the generating function is

$$r(x) = \sum_{n=1}^{\infty} n x^{2n} + \sum_{n=1}^{\infty} n x^{2n+1} = q(x) , \tag{30}$$

as required.

Now suppose that

$$V_0\left(\sum_{n=1}^{\infty} \lambda_n a_{-n}a_{-n}|0\rangle\right) = 0 . \tag{31}$$

Then, (10) gives

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\lambda_n}{((n-1)!)^2} \sum_p (p+1)(p+2)\cdots(p+n-1) && (32) \\ &\qquad \cdot(-p+1)(-p+2)\cdots(-p+n-1) : a_p a_{-p} : + \lambda_1 \sum_p : a_p a_{-p} := 0 . && (33) \end{aligned}$$

Considering the $p = 1$ terms gives us $\lambda_1 = 0$, while the terms for $p > 1$ give

$$\lambda_1 + \sum_{n=2}^{p} \lambda_n (-1)^{n-1} \binom{p+n-1}{n-1} \binom{p-1}{n-1} , \tag{34}$$

and so we deduce that $\lambda_n = 0$ for all n as required.

Note that the trick in the choice of basis states is to choose all the oscillator levels to go off to infinity as the level of the state does. If we had chosen, as in the above, at least one a_{-1}, then every λ_n would have been involved in every term $a_p a_{-p}$, leading to an uncontrollable infinite set of equations. The same idea however cannot be applied to states involving more than two oscillators.

For the states $a_{-1}a_{-n}a_{-n}|0\rangle$, the above technique fails, as arbitrarily large conformal weight states contribute to all the operators. So, to prove that the method cannot generalize beyond two oscillators, we only have to show that $a_{-1}a_{-n}a_{-n}|0\rangle$ is not an L_{-1} descendant (this will also be part of the required proof to show that the above set of states is a suitable basis if we wished to proceed with the method). The proof is by contradiction, exactly as in (26-28).

Let us instead take a more general approach, though this reveals less about the explicit structures involved, and consider a quasi-primary state ψ such that $V_{-1}(\psi) = 0$. We wish to show that $\psi = \lambda|0\rangle$ for some $\lambda \in \mathbf{C}$. Now, since ψ is quasi-primary, [8]

$$[L_n, V(\psi, z)] = z^n \left(z\frac{d}{dz} + (n+1)L_0 \right) V(\psi, z)\,, \tag{35}$$

for $n = \pm 1$, giving

$$[L_1, V_{-1}(\psi)] = V_0(L_0\psi) \tag{36}$$

and

$$[L_{-1}, V_0(L_0\psi, z)] = -V_{-1}((L_0 - 1)L_0\psi)\,. \tag{37}$$

Since $(L_0 - 1)L_0\psi$ is still quasi-primary, we may repeat the procedure, and find that

$$V_0\left(((L_0-1)L_0)^n L_0\psi\right) = V_{-1}\left(((L_0-1)L_0)^n \psi\right) = 0\,, \tag{38}$$

for all $n \geq 0$. For $\psi = \sum_n \psi_n$ where $L_0\psi_n = n\psi_n$, this gives $V_0(\psi_n) = V_{-1}(\psi_n) = 0$ for $n > 1$.

This now reduces the problem at each level to a finite one, and hence should render it tractable. For example, the first non-trivial level for the product of three creation oscillators in the vacuum sector, as we discussed above, is at conformal weight 5. A general state can be written as

$$\psi_5 = \lambda_{113} a_{-1}a_{-1}a_{-3}|0\rangle + \lambda_{122} a_{-1}a_{-2}a_{-2}|0\rangle\,. \tag{39}$$

Now

$$V(\psi_5)_0 = \frac{\lambda_{113}}{2} \sum_{p+q+r=0} (r+1)(r+2) : a_p a_q a_r : + \lambda_{122} \sum_{p+q+r=0} (q+1)(r+1) : a_p a_q a_r :\,, \tag{40}$$

and so we see, from considering the terms in $a_{-1}a_{-1}a_2$, that $\lambda_{113}=0$ (and hence also that $\lambda_{122}=0$). Thus, $\psi_5=0$ as required.

In general, consider a state in the vacuum sector containing n creation operators, say

$$\psi=\sum_{\{ij\ldots k\}}\lambda_{ij\ldots k}a_{-i}a_{-j}\cdots a_{-k}|0\rangle\,, \tag{41}$$

where we have ordered $i\le j\le\cdots\le k$. We say that $\lambda_{ij\ldots k}>\lambda_{lm\ldots n}$ if $i>l$ or ($i=l$ and $j>m$), and so on. Then we take the λ's in ascending order, and for $\lambda_{ij\ldots kl}$ consider the term of $V_0(\psi)$ in $a_{-i}a_{-j}\cdots a_{-k}a_{i+j+\cdots+k}$.

Now, we know from the form of the vertex operators that the contribution from the creation operator a_{-i} in ψ to a_r vanishes if $1\le r\le i-1$, and we see that we can deduce in turn that all the $\lambda_{ij\ldots k}$, and hence ψ itself, must vanish, as required.

For $\psi=\prod a|\lambda\rangle$ (using an obvious schematic notation), $V(\psi)_n$ is

$$\sum_m V_m\left(\prod_{a\perp\lambda}a|0\rangle\right)V_{n-m}(\text{remainder})\,, \tag{42}$$

and we reduce our considerations to one-dimension as above (*i.e.* either $V_0\left(\prod_{a\perp\lambda}a|0\rangle\right)=0$ or $V_0(\text{remainder})=0$).

[Note that we consistently are using the fact that V(subspace)$\subset$subspace (of the vector space spanned by Λ), and that operators associated with these subspaces commute (normal ordering of (42) is unnecessary).]

Now, suppose that $V_0(\phi|\lambda\rangle)=0$ for ϕ a combination of creation operators (in the direction of λ) of weight N. We wish to show that $\phi=0$. Consider the form of the vertex operator. The 0'th mode acting in the sector with momentum $r\lambda$ is given by

$$\sum_{\substack{m\ge0\\ p\ge0}}Y_mV_{n+m-p}(\phi)X_p\,, \tag{43}$$

where $n=(r+\frac{1}{2})\lambda^2$ and X_p and Y_m are appropriate modes of $e^{\mp\lambda\cdot\sum_{s>0}\frac{a_{\pm s}}{s}}$.

Consider first the term involving a single oscillator (a_n). This is

$$(V_n(\phi)|_{a_0=r\lambda})_{\text{single-oscillator}}+(-1)^N\frac{\lambda\cdot a_n}{n}\sum_m(-r\lambda)^m\,, \tag{44}$$

where the sum over m is over the terms in ϕ formed from a product of m creation operators. (Note that the a_0 term in $V(a_{-t}|0\rangle,z)$ is $z^{-t}(-1)^{t-1}a_0$.)

Naively, we could vary r and effectively get vanishing of arbitrarily large modes of the operator part of the vertex operator, which should give vanishing of the state. However, the contribution to terms with lower numbers of oscillators from powers of a_0 occurring in higher states is difficult to handle, and in any case we can instead use a very simple argument. Consider just the case $r = 0$. Then

$$\left(V_{\frac{1}{2}\lambda^2}(\phi)|_{a_0=0}\right)_{\text{single-oscillator}} = 0\,, \tag{45}$$

and we deduce that the single-oscillator term in ϕ (*i.e.* a_{-N}) vanishes. Then we consider the two-oscillator terms. There is no contribution from X and Y (other than 1!), since the single-oscillator term in ϕ vanishes. The result for the zero-momentum sector which we have already proven tells us that these vanish (the vanishing of $V_{\frac{1}{2}\lambda^2}(\phi)$ implies, from the L_{-1} commutation relation, that $V_0\left(L_0(L_0+1)\cdots(L_0+\frac{1}{2}\lambda^2-1)\phi\right)$ vanishes). Proceeding in this way, we deduce that $\phi = 0$, as required.

5 Determinism of the theories $\widetilde{\mathcal{H}}(\Lambda)$

We must show that vanishing of

$$\mathcal{V}\left((\psi,\chi)\right)_0 = \begin{pmatrix} V(\psi)_0 & \overline{W}(\chi)_0 \\ W(\chi)_0 & V_T(\psi)_0 \end{pmatrix} \tag{46}$$

implies vanishing of ψ and χ (under suitable restrictions on the states). For the state ψ, the results of the previous section are sufficient (note that we do not even have to consider the structure of the twisted vertex operators V_T). However, the twisted structure enters into consideration of the state χ.

It should be noted that it is not possible to reduce to a one-dimensional problem as we did in the previous section. Consider, for example,

$$\overline{W}(c^i_{-\frac{1}{2}}\chi, z)c^j_{-\frac{1}{2}}\chi\,, \tag{47}$$

where $i \neq j$ and χ is in the spinor ground state. If this contains negative powers of z, then the operator product expansion (2) of $\mathcal{V}(c^i_{-\frac{1}{2}}\chi, z)$ and $\mathcal{V}(c^j_{-\frac{1}{2}}\chi, w)$ has singular terms and so leads to a non-zero commutator on contour integration [7], unlike in the untwisted case, and we can no longer treat the orthogonal (vectorial) subspaces independently. Now, (47) is

$$\sum_n \overline{W}_n(c^i_{-\frac{1}{2}}\chi)z^{-n-\frac{1}{2}-\frac{d}{16}}c^j_{-\frac{1}{2}}\chi. \tag{48}$$

Since the state resulting from the action of the n'th mode has weight $\frac{d}{16}+\frac{1}{2}-n$, then for negative powers of z we simply require a non-zero piece in (47) of weight $\leq \frac{d}{8}$. Putting in the explicit expression for $\overline{W}$, we see this is clearly true, even in the case $d = 8$ (at least if there exist vectors of length squared two in the lattice). Thus, the twisted sector mixes up the independent dimensions of the untwisted sector, and is implicitly tied up with the momenta (*c.f.* $V(a^i_{-1}|0\rangle, z)$ and $V(|\lambda\rangle, w)$ do not commute for $\lambda^i \neq 0$).

Take χ to be of conformal weight N, as we did for the untwisted state in the previous section. Now, using the explicit results of [3],

$$\begin{aligned} \overline{W}(\overline{\chi}, z)\chi &= W\left(e^{z^* L_1} z^{*-2L_0}\chi, 1/z^*\right)^\dagger \chi & (49) \\ &= \sum_{\lambda\in\Lambda} \langle\chi|\gamma_\lambda{}^\dagger z^{-2L_0} e^{zL_{-1}} : e^{B(1/z^*)^\dagger} : e^{\frac{1}{z}L_1}|\chi\rangle e^{A(1/z^*)^\dagger}|\lambda\rangle\,, & (50) \end{aligned}$$

where $A(w)$ and $B(w)$ are of the form $A(w) = Ap^2 \log w + \sum_{n,m\geq 0} A_{mn} a_m \cdot a_n w^{-n-m}$ and $B(w) = \sum_{n\geq 0,s} B_{ns} a_n \cdot c_s w^{-n-s}$. But $\overline{W}(\overline{\chi})_0\chi$ is the weight N piece of this, and so if we consider a term in the sector with momentum λ for $\lambda^2 = 2N$ (we must check that such an assignment is possible – see below for a discussion), then we must have vanishing of

$$\langle\chi|\gamma_\lambda{}^\dagger e^{L_{-1}} e^{B^\dagger} e^{B} e^{L_1}|\chi\rangle\,, \tag{51}$$

where

$$B = -\sum_{\substack{s>0 \\ s\in\mathbf{Z}+\frac{1}{2}}} \frac{\lambda\cdot c_s}{s}\,. \tag{52}$$

Adjust the ground state spinor, if necessary, of one of the χ states so that the ground state matrix element is non-zero (*e.g.* multiply it by γ_λ), and then we see that

$$||e^B e^{L_1}|\chi\rangle|| = 0\,, \tag{53}$$

and hence that $\chi = 0$, as required.

Now we consider whether it is possible to choose $\lambda \in \Lambda$ such that $\lambda^2 = 2N$ for any $N \in \mathbf{Z}_+$. We have the restriction that $\sqrt{2}\Lambda^*$ be even (so that the twisted conformal field theory is consistent), but even self-duality is clearly not sufficient. For example, of the 24 even self-dual lattices in 24 dimensions, it is possible to make such a choice for all but one of the lattices, namely the Leech lattice. As is well known, the Leech lattice has no vectors of length squared two. But this corresponds to conformal weight one, and the lowest weight in the twisted sector after projection is two, so there is in fact no problem in this case.

The general situation remains to be understood. However, note that the minimal norm for an even self-dual (Type II) lattice [1] is $2\left[\frac{d}{24}\right]+2$, so that if the coefficients in the theta series of the lattice are all non-zero above the minimal norm then we have the required result. In any particular case, this is trivial to check, and even if the result does not hold, we can always choose a λ such that λ^2 lies within the minimal norm of $2N$, and we have to simply extend the above analysis by considering the appropriate number of terms in the expansion of $e^{A(1/z^*)^\dagger}$.

6 Conclusions

We have proposed a condition which will guarantee that any (inner) continuous automorphism of a bosonic meromorphic hermitian conformal field theory is generated by the zero modes of the vertex operators corresponding to the states of conformal weight one, and checked that this is a reasonable condition to impose on a conformal field theory by showing that it holds for the FKS theories $\mathcal{H}(\Lambda)$. The proof that the twisted theories $\widetilde{\mathcal{H}}(\Lambda)$ are deterministic depends on the theta function of the lattice being suitably behaved, though it is clear that a (more intricate) proof not relying on such assumptions can be found. In any case, the proof holds for the known self-dual theories at central charge 24 or less, and in particular is consistent with Frenkel, Lepowsky and Meurman's result [5] that there are no continuous automorphisms of the Monster conformal field theory.

A Equivalence of definitions of determinism

Suppose $\mathcal{H}$ is such that $V_0(\phi)=0$ and ϕ quasi-primary implies that ϕ is of weight one (the same statement as that in the text by linearity of the vertex operators in their arguments). Then suppose that $V(\psi)_{-1}=0$. If ψ is quasi-primary, the usual commutation relation with L_1 [8] gives $V_0(L_0\psi)=0$, and so $L_0\psi$, by the assumption, is of weight one (clearly being quasi-primary). Hence $\psi=\lambda|0\rangle+\chi$, for some $\lambda\in\mathbf{C}$ and χ of conformal weight one. But $V_{-1}(\lambda|0\rangle+\chi)|0\rangle=\chi$, and so $V_{-1}(\psi)=0$ implies $\chi=0$, and we have the required result.

Conversely, suppose that $V_{-1}(\psi)=0$ and ψ quasi-primary implies that $\psi=\lambda|0\rangle$ for some $\lambda\in\mathbf{C}$. Then, if $V_0(\phi)=0$, $[L_{-1},V_0(\phi)]=0$, *i.e.* $V((L_0-1)\phi)_{-1}=0$. If ϕ is quasi-primary, then so is $(L_0-1)\phi$, and so we have $(L_0-1)\phi=\lambda|0\rangle$ for some $\lambda\in\mathbf{C}$, *i.e.* $\phi=-\lambda|0\rangle+\chi$, for some state χ of weight one. But $V_0(\phi)=0$, and so $\lambda=0$. Thus, we have the required equivalence.

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